*** An excellent solution was received from Professor Zerr. He takes the horizontal line on which the wheel travels as the x-axis, and gets for the equation of the path of the fly,

$$x=a\theta\left(1-\frac{\sin\theta}{2n\pi}\right),\ y=a\left(1-\frac{\theta\cos\theta}{2n\pi}\right), \text{ for required length.}$$

$$s=a\int_0^{2\pi/m}\sqrt{1+\frac{1+\theta^2}{4\pi^2n^2}-\frac{\sin\theta+\theta\cos\theta}{\pi n}}\,d\theta. \qquad \text{F.}$$

192. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Show that the volume V of the hyper-ellipsoid with semi-axes a_1, a_2, a_3, a_4, etc., in space of $2n$ and $2n+1$ dimensions is

$$V_{2n}=\frac{a_1.a_2.a_3.....a_{2n}.\pi^n}{1.2.3.4.5..........n};\qquad V_{2n+1}=\frac{2^{n+1}.a_1.a_2.a_3\,.........\,a_{2n+1}.\pi^n}{1.3.5.7.9..............(2n+1)}.$$

Solution by the PROPOSER.

Let $\left(\frac{x_1}{a_1}\right)^2+\left(\frac{x_2}{a_2}\right)^2+\left(\frac{x_3}{a_3}\right)^2+.....+\left(\frac{x_r}{a_r}\right)^2=1$ be the equation to the hyper-ellipsoid. Then its volume is $V=2^r\int\int\int.....dx_1\,dx_2\,dx_3.....dx_r$.

Let $x_1/a_1=y_1$, $x_2/a_2=y_2$, ..., $x_r/a_r=y_r$.

$\therefore V=2^ra_1a_2a_3.....a_r\int\int\int.....dy_1\,dy_2\,dy_3.....dy_r$, subject to the condition, $y_1^2+y_2^2+y_3^2+.....+y_r^2=1$.

$\therefore V=\dfrac{a_1a_2a_3.....a_n[\Gamma(\frac{1}{2})]^r}{\Gamma(1+\frac{1}{2}r)}$.

When $r=2n$,

$$V=\frac{a_1a_2a_3.....a_{2n}\,\pi^n}{1.2.3.4.....n}.$$

When $r=2n+1$,

$$V=\frac{a_1a_2a_3.....a_{2n+1}\,\pi^n\,\Gamma(\frac{1}{2})}{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}.....\frac{2n+1}{2}\Gamma(\frac{1}{2})}=\frac{2^{n+1}a_1a_2a_3.....a_{2n+1}\,\pi^n}{1.3.5.7.9.....(2n+1)}.$$

193. Proposed by F. P. MATZ, Sc. D., Ph. D., Reading, Pa.

Find the eccentricity of the maximum semi-ellipse inscribed in a given isosceles triangle.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va., and J. SCHEFFER, Hagerstown, Md.

Let the mid-point of the base be the origin, a=altitude, b=base of triangle. Let $x^2/m^2+y^2/n^2=1$ be the ellipse. Then πmn=maximum.

$\therefore n/m=y/x$.

Let (h, k) be the tangent point of the ellipse with a side.

Then $a=n^2/k$, $\frac{1}{2}b=m^2/n$, or $k/h=2a/b=n/m$. Also $2ah+bk=hk$.

$\therefore k=\frac{1}{2}a$, $h=\frac{1}{4}b$. $\therefore b^2n^2+4a^2m^2=16m^2n^2$. And $bd=2am$.

$\therefore m=\dfrac{b}{2\sqrt{2}}$, $n=\dfrac{a}{\sqrt{2}}$; $(\text{eccentricity})^2=\dfrac{4a^2-b^2}{4a^2}$.

II. Solution by A. H. HOLMES, Brunswick, Me.

Let $2a$=base of the isosceles triangle, and b its perpendicular height.

Construct on $2a$ an equilateral triangle, and inscribe in it a semi-circle its diameter collinear with base $2a$. Then the radius of the semi-circle will be $\dfrac{a\sqrt{3}}{2}$ which is one-half the perpendicular of the equilateral triangle. Now consider this triangle to be projected into an isosceles triangle whose base will be, of course, the same as that of the equilateral triangle, but whose perpendicular height is b. The semi-circle inscribed in the equilateral triangle will be projected into the maximum semi-ellipse that can be inscribed in the isosceles triangle, and one of its semi-axes will have the same proportion to the perpendicular of the isosceles triangle that the radius of the semi-circle has to the perpendicular of the equilateral triangle.

$\therefore$ Eccentricity of ellipse$=\dfrac{\sqrt{(b^2-3a^2)}}{b}$ or $\dfrac{\sqrt{(3a^2-b^2)}}{a\sqrt{3}}$, accordingly as $\sqrt{(a^2+b^2)}$ is greater or less than $2a$. If b=one of the sides,

$$\varepsilon=\sqrt{\frac{b^2-4a}{b^2-a^2}}, \text{ or } \frac{\sqrt{(4a^2-b^2)}}{a\sqrt{3}}.$$

Also solved by Jacob Westlund.

DIOPHANTINE ANALYSIS.

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123. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Of two numbers $a_ib_ic_id_ie_i$ $(i=1, 2)$ it is given that their 10 digits a_1, ..., e_2 form a permutation of 0, 1, ..., 9, and that the sum of the two is $x3951$. Give an immediate evaluation of x; also list the possible pairs a_1, a_2 ; ..... ; e_1, e_2.

Solution by the PROPOSER.

Since the sum of the 10 digits is 45, $x+18$ must be a multiple of 9 by the rule of casting out of 9's. Hence $x=9$.

Next, on adding the third column there cannot be 1 to carry; otherwise c_1+c_2 or c_1+c_2+1 would be 19, and $c_1 \overline{\gtrless} 9$, $c_2 \overline{\gtrless} 9$. Hence

$$(1)\quad b_1+b_2=3,\ a_1+a_2=9;\ \text{or}\ (2)\quad b_1+b_2=13,\ a_1+a_2=8.$$

If $e_1+e_2=1$, the b's are not 0, 3; nor 1, 2. Hence in this case,

$e_1, e_2=0, 1$; $b_1+b_2=13$; $a_1+a_2=8$; $d_1+d_2=5$, $c_1+c_2=9$; or $d_1+d_2=15$, $c_1+c_2=8$.